TOPOLOGIES FOR ALMOST CONTINUOUS FUNCTIONS

by

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "TOPOLOGIES FOR ALMOST CONTINUOUS FUNCTIONS", submitted by ERIC D. J. BUCKLEY in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

In his paper [10] on almost continuous mappings, T. Husain makes the following definition; let f be a mapping of a Hausdorff topological space (X,η) into a Hausdorff topological space (Y,ρ) : then f is said to be almost continuous at $x \in X$ if for each neighbourhood V of $f(x)$ in Y, $\overline{f^{-1}(V)}$ is a neighbourhood of x. A function f is said to be almost continuous on X if it is so at every $x \in X$.

J. Stallings, in [15], defined an almost continuous function f in terms of its graph $\Gamma(f) = \{(x, f(x)) | x \in X\} \subset X \times Y$ as follows: a function f from a space (X,η) to a space (Y,ρ) is almost continuous if for every open set $N \subset X \times Y$ (with product topology) containing $\Gamma(f)$, there exists a continuous function g on X to Y such that $\Gamma(g) \subset N$.

If we let $X = Y$ be the real numbers with usual topology and define the function $g:X \to Y$ by $g(x) = \sin \frac{1}{x}$, $x \neq 0$, $g(0) = 0$, then g is almost continuous by Stallings' definition, but not by Husain's, whereas if $X = Y = [0,1]$ with the subspace topology of real numbers, and $f:X \to Y$ is defined by $f(x) = 0$; for rational x, and $f(x) = 1$ for irrational x, then f is almost continuous by Husain's definition, but not by that of Stallings.

Since neither definition implies the other we shall refer to functions satisfying Husain's definition as H-functions, and those satisfying Stallings' definition as S-functions. The symbols $\underline{H}$ and

$\underline{S}$ will be used to denote the families of H-functions and S-functions respectively on the space (X, η) to (Y, ρ) where neither X nor Y is assumed to be Hausdorff. Following common convention, the symbols $\underline{C}$ and Y^X will denote the family of all continuous functions on X to Y, and $\underline{F}$ the family of all functions on X to Y.

A topology, Γ, called the graph topology, was introduced by Naimpally [12] on the set $\underline{F}$, having the property that $\underline{S}$ is a closed subset of $(\underline{F}, \Gamma)$. It is the purpose of this thesis to construct a new topology τ for $\underline{F}$ in such a way that $\underline{H}$ becomes a closed subset of $(\underline{F}, \tau)$, and to compare this topology with some known function space topologies.

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I dedicate this thesis to my wife and children, with a special thanks for their understanding and lack of complaint for the many times when it was necessary for them to make certain sacrifices to the continuation of my studies.

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CHAPTER I

A NEW FUNCTION SPACE TOPOLOGY

1.1 Definition. Let (X, η) and (Y, ρ) be topological spaces. A function f on X to Y is said to be an H-function, or $f \in \underline{H}$, if for every $x \in X$ and every neighbourhood U of $f(x)$ in Y, the set $\overline{f^{-1}(U)}$ is a neighbourhood of x.

A topology τ on $\underline{F}$ can be described in terms of convergence by specifying what is to be understood as the τ-convergence of a net in $\underline{F}$. (Kelley, [11], pp. 65-66). We therefore make the following definition.

1.2 Definition. Let $(D, \geq)$ be a directed set and $\{f_n : n \in D\}$ a net in $\underline{F}$. The net $\{f_n\}$ is said to be τ-convergent to $f \in \underline{F}$, in symbols $f_n \xrightarrow{\tau} f$, if and only if :

i) for each $x \in X$, the net $\{f_n(x)\}$ converges in Y to $f(x)$; and

ii) for every neighbourhood U of $f(x)$ in Y, there exists $n_o \in D$ such that if $n \geq n_o$ then $\overline{f_n^{-1}(U)} \subset \overline{f^{-1}(U)}$.

A set $A \subset \underline{F}$ will be called τ-open if and only if no net in the complement of A is τ-convergent to an element of A, and the symbol τ will thus be used as well to denote the family of all τ-open

subsets of $\underline{F}$. Then the pair $(\underline{F},\tau)$ is a topological space. (It must first be established that there does exist a topology for $\underline{F}$ such that the convergence of Definition 1.2 is convergence relative to that topology; see Kelley [11], pp. 73-74. This will be proved below by Theorem 1.7 and is therefore assumed here.)

<u>1.3 Theorem</u>: The set $\underline{H}$ is closed in $(\underline{F},\tau)$.

Proof: Let $\{f_n\}$ be a net in $\underline{H}$ such that $f_n \xrightarrow{\tau} f$. Let $x \in X$ and U a neighbourhood of $f(x)$. For each n, $\overline{f_n^{-1}(U)}$ is a neighbourhood of x , so there exists an open set $G_n \subset X$ such that $x \in G_n \subset \overline{f_n^{-1}(U)}$. There exists n_o such that $\overline{f_n^{-1}(U)} \subset \overline{f^{-1}(U)}$ for all $n \geq n_o$. Thus $x \in G_{n_o} \subset \overline{f_{n_o}^{-1}(U)} \subset \overline{f^{-1}(U)}$ and hence $\overline{f^{-1}(U)}$ is a neighbourhood of x. Therefore $f \in \underline{H}$ so $\underline{H}$ is closed in $(\underline{F},\tau)$ by Theorem 2-C of Kelley [11], page 66, completing the proof.

The pointwise convergence topology, p.c., for $\underline{F}$ is that topology having as subbase the family of all sets of the form $\{f \in \underline{F} \mid f(x) \in U$, $x \in X$, U open in $Y\} = W(x,U)$. An obvious result of Definition 1.2 is the following theorem.

<u>1.4 Theorem</u>: The pointwise convergence topology for $\underline{F}$ is contained in τ .

<u>1.5 Theorem</u>: The evaluation maps e_x from $(\underline{F},\tau)$ to (Y,ρ) defined by $e_x(f) = f(x)$ are continuous for every $x \in X$.

Proof: The pointwise convergence topology for $\underline{F}$ is the smallest topology for which the evaluation maps are continuous. (Pervin [13], p. 147.)

By theorem 1.4 they are also continuous relative to τ, completing the proof.

Given an element $f \in (\underline{F}, \tau)$ it is desirable to construct finitely many open sets containing f, whose intersection is also open, i.e. to construct a subbase for the topology τ. Let $f \in \underline{F}$ be chosen, and let $x \in X$ such that U is an open neighbourhood of $f(x)$. We denote by the symbol $[f,x,U]$ the set $\{g \in W(x,U) \mid \overline{g^{-1}(U)} \subset \overline{f^{-1}(U)}\}$, where the notation illustrates that membership depends on all of f, x and U. Clearly we have $f \in [f,x,U]$.

<u>1.6 Theorem</u>. $[f,x,U]$ is open in $(\underline{F}, \tau)$.

Proof: Suppose $[f,x,U]^c$ is not closed. Then there exists a net $\{f_n \mid n \in D\}$ in $[f,x,U]^c$ such that $\{f_n\} \xrightarrow{\tau} g \in [f,x,U]$. Then we have $g(x) \in U$ and $\overline{g^{-1}(U)} \subset \overline{f^{-1}(U)}$. There exists $n_o \in D$ such that $\overline{f_n^{-1}(U)} \subset \overline{g^{-1}(U)}$ for $n \geq n_o$. Therefore f_n must fail to be in $W(x,U)$ for all $n \geq n_o$. Let $D' = \{n' \in D \mid n' \geq n_o\}$. Then $\{f_{n'} \mid n' \in D'\}$ is a subnet of $\{f_n \mid n \in D\}$ which also converges to g. Thus $f_{n'}(x) \in U^c$ for all $n' \in D'$ since $f_{n'}$ fails to be in $W(x,U)$. Since U^c is closed and $\{f_{n'}(x)\}$ is a net in U^c which converges pointwise to $g(x)$, it follows that $g(x) \in U^c$, contradicting the supposition that $g \in [f,x,U]$. Therefore no net in $[f,x,U]^c$ can converge to a point in $[f,x,U]$, so $[f,x,U]$ is open in τ.

It will now be shown that a subbase for τ consists of all sets of the form $[f,x,U]$ defined above. Let π be the family of all

unions of sets which are finite intersections of sets of the form $[f,x,U]$ for $f \in \underline{F}$, $x \in X$, $f(x) \in U$ open in Y. Then π is a topology for $\underline{F}$, and it will be shown that $\pi = \tau$.

<u>1.7 Theorem:</u> A net $\{f_n\}$ in $\underline{F}$ is τ-convergent to $f \in \underline{F}$ if and only if it converges to f in $(\underline{F}, \pi)$.

Proof: Let $\{f_n\}$ be a net in $\underline{F}$ such that $f_n \xrightarrow{\tau} f$. Let W be any element of π with $f \in W$. There exist finitely many sets $[f,x_i,U_i]$, $i = 1,2,\ldots,k$ containing f, such that $f \in \bigcap_{i=1}^{k} [f,x_i,U_i] \subset W$. For each i, the set $[f,x_i,U_i]$ is open in τ and $f \in [f,x_i,U_i]$, so there exists n_i such that $f_n \in [f,x_i,U_i]$ for $n \geq n_i$. Let $n_o \geq n_1,n_2,\ldots,n_k$. Then $f_n \in \bigcap_{i=1}^{k} [f,x_i,U_i] \subset W$ for $n \geq n_o$. Thus $\{f_n\}$ converges to f in the space $(\underline{F},\pi)$. Conversely, if $\{f_n\}$ converges to f in the topology π, then given U open with $f(x) \in U$ the next $\{f_n\}$ is eventually in $[f,x,U]$. Thus $f_n(x)$ is eventually in U and $\overline{f_n^{-1}(U)}$ is eventually a subset of $\overline{f^{-1}(U)}$. So by Definition 1.2 we have $f_n \xrightarrow{\tau} f$, completing the proof.

<u>1.8 Corollary:</u> $\tau = \pi$, thus $\{[f,x,U] \mid f \in \underline{F},\ f(x) \in U \text{ open}\}$ is a subbase for τ.

Proof: A topology is uniquely determined by convergence of nets. (Kelley [11], p. 65.)

CHAPTER II

PROPERTIES OF ALMOST CONTINUOUS FUNCTIONS

In this chapter we shall discuss the conditions under which the properties of S-almost continuity and H-almost continuity are preserved under composition and restriction of mappings. Certain results of Stallings [15] are included for completeness of exposition.

2.1 Theorem: Let X, Y and Z be topological spaces, and let $f: X \to Y$ be an S-function, and $g: Y \to Z$ continuous. Then $gf: X \to Z$ is an S-function.

Proof: Let N be open in $X \times Z$ such that $\Gamma(gf) \subset N$. Define $\bar{g}: X \times Y \to X \times Z$ by $\bar{g}(x,y) = (x, g(y))$. Then $\bar{g}$ is continuous, hence $\bar{g}^{-1}(N)$ is open in $X \times Y$. Let $(x,y) \in \Gamma(f)$. Then $\bar{g}(x,y) = (x, g(y)) = (x, gf(x)) \in \Gamma(gf) \subset N$. But $(x,y) \in \bar{g}^{-1}\,\bar{g}(x,y) \subset \bar{g}^{-1}(N)$ so $\Gamma(f) \subset \bar{g}^{-1}(N)$. Thus there exists a continuous function $F: X \to Y$ such that $\Gamma(F) \subset \bar{g}^{-1}(N)$. Now $gF: X \to Z$ is continuous and $\Gamma(gF) \subset N$: for if $(x,z) \in \Gamma(gF)$ we have $(x,z) = (x, gF(x) = \bar{g}(x, F(x)) \subset \bar{g}(\Gamma(F)) \subset \bar{g}\,\bar{g}^{-1}(N) \subset N$. Therefore gf is an S-function.

2.2 Theorem: Let $f: X \to Y$ be an S-function and C a closed subset of X. Then f/C is an S-function.

Proof: Let N be open in $C \times Y$ such that $\Gamma(f|C) \subset N$. There exists an open set $N^* \subset X \times Y$ such that $N = N^* \cap (C \times Y)$. The set $U = N^* \cup (X - C \times Y)$ is open in $X \times Y$. If $x \in C$ then $(x, f(x)) = (x, f|C(x)) \in \Gamma(f|C) \subset U$ and if $x \notin C$ then $(x, f(x)) \in X - C \times Y$. Thus $\Gamma(f) \subset U$, and since f is an S-function there exists a continuous function $F: X \to Y$ such that $\Gamma(F) \subset U$. Now $F|C$ is continuous, and $\Gamma(F|C) \subset N$; for if $(x, y) \in \Gamma(F|C)$ then $x \in C$ and $y = F(x)$ so $(x, y) \in \Gamma(F) \subset U$. Thus $(x, y) \in N^* \subset N$. Therefore $f|C$ is an S-function.

A topological space is said to be T_2 or Hausdorff, if distinct points are contained in disjoint open sets.

2.3 Theorem: Let X be a compact Hausdorff space, Y a Hausdorff space, and Z any topological space. If $f: X \to Y$ is continuous and $g: Y \to Z$ an S-function, then $gf: X \to Z$ is an S-function.

Proof: Since X is compact and f continuous then $f(X)$ is closed in Y. Thus $g|f(X)$ is an S-function by Theorem 2.2. We may therefore assume without loss of generality that f maps X onto Y.

Let N be open in $X \times Z$ with $\Gamma(gf) \subset N$. Define $f_*: X \times Z \to Y \times Z$ by $f_*(x, z) = (f(x), z)$. Then f_* is continuous and $f_*(\Gamma(gf)) = \{(f(x), gf(x)) | x \in X\} = \Gamma(g)$. Since Y is Hausdorff, singletons are closed in Y, hence $f^{-1}(y)$ is closed, and therefore compact in X for any $y \in Y$. Let $y \in Y$, and for every $x \in f^{-1}(y)$ let N_x be an open neighbourhood of x and M_x an open neighbourhood of $gf(x) = g(y)$ such that $N_x \times M_x \subset N$. Let $N_{x_1}, N_{x_2}, \ldots, N_{x_k}$ be a finite cover of

$f^{-1}(y)$ and $M_{x_1}, M_{x_2}, \ldots, M_{x_k}$ the corresponding open sets in Z. Now $g(y) \in M_{x_j}$ for each $j = 1,2,\ldots,k$, and $f^{-1}(y) \subset \bigcup_{i=1}^{k} N_{x_i}$ so $X - f^{-1}(y) \supset X - \bigcup_{i=1}^{k} N_{x_i}$. Hence $f[X-f^{-1}(y)] \supset f[X - \bigcup_{i=1}^{k} N_{x_i}]$, or $Y - f[X - \bigcup_{i=1}^{k} N_{x_i}] \subset Y - f[X - f^{-1}(y)]$. Let $U_y = Y - f[X - \bigcup_{i=1}^{k} N_{x_i}]$. Then U_y is open in Y. Let $W_y = U_y \times \bigcap_{i=1}^{k} M_{x_i}$ open in $Y \times Z$. Then W_y is a neighbourhood of $(y, g(y))$: for $g(y) \in \bigcap_{i=1}^{k} M_{x_i}$ and for each x_i we have $x_i \in f^{-1}(y)$ so $f(x_i) = y \in f(\bigcup_{i=1}^{k} N_{x_i})$. Thus $y \notin f[X - \bigcup_{i=1}^{k} N_{x_i}]$ or $y \in U_y = Y - f[X - \bigcup_{i=1}^{k} N_{x_i}]$. Now $f_*^{-1}(W_y) = \{(x,z) | (f(x), gf(x)) \in W_y\} \subset \Gamma(gf) \subset N$. Let $W = \bigcup_{y \in Y} W_y$. Then W is an open set containing $\Gamma(g)$, thus there exists a continuous function $G: Y \to Z$ such that $\Gamma(G) \subset W$. Then Gf is continuous and $f_*(\Gamma(Gf)) = f_*\{(x, Gf(x)) | x \in X\} = \{(f(x), Gf(x)) | x \in X\} = \Gamma(G) \subset W$. Hence $\Gamma(Gf) \subset f_*^{-1}(W) = f_*^{-1}(\bigcup_{y \in Y} W_y) = \bigcup_{y \in Y} f_*^{-1}(W_y) \subset N$. Therefore Gf is a continuous function such that $\Gamma(Gf) \subset N$, so gf is an S-function.

<u>2.4 Theorem</u>: Let X, Y, Z be topological spaces. If $f: X \to Y$ is an H-function and $g: Y \to Z$ is continuous, then $gf: X \to Z$ is an H-function.

Proof: Let $x \in X$ with $gf(x) \in U$ open in Z. There exists an open neighbourhood V of $f(x)$ such that $V \subset g^{-1}(U)$. Thus $f^{-1}(V) \subset f^{-1}g^{-1}(U) = (gf)^{-1}(U)$ so $f^{-1}(V) \subset (gf)^{-1}(U)$. Since $f \in \underline{H}$, there exists an open neighbourhood G of x such that $G \subset f^{-1}(V)$. Therefore $x \in G \subset (gf)^{-1}(U)$ so gf is an H-function.

It remains an open question whether, or under what conditions,

the conclusion of Theorem 2.4 holds when f is a continuous function and g is an H-function. If, however, both f, g are H-functions but not continuous, then the composite function need not be an H-function, as in the following example.

<u>2.5 Example</u>: Let $X = Y = Z$ be the space of real numbers with usual open interval topology. Define the functions f and g by

$$f(x) = \begin{cases} 2 + x\,; & \text{if } x \text{ is rational} \\ -2 + x\,; & \text{if } x \text{ is irrational} \end{cases}$$

$$g(x) = \begin{cases} -2 + x\,; & \text{if } x \text{ is rational and } x \neq 2 \\ 2 + x\,; & \text{if } x \text{ is irrational or } x = 2 \end{cases} .$$

Then f and g are both H-functions on the Reals to the Reals. However, the composite function gf has definition

$$gf(x) = \begin{cases} x\,; & \text{if } x \neq 0 \\ 4\,; & \text{if } x = 0 \end{cases}$$

which is continuous everywhere except at zero. The interval $(3,5)$ is an open neighbourhood of $gf(0)$ and $(gf)^{-1}((3,5)) = \{0\} \cup [3,5]$ which is not a neighbourhood of zero. Hence gf fails to be an H-function at zero.

We conclude this chapter with a counter-example showing that the restriction of an H-function to a subset of its domain need not be an H-function.

<u>2.6 Example</u>: Let $X = Y$ be the real number space; let A be the set of rationals and I the set of integers. Define the function $f: X \to Y$ as follows:

$$f(x) = \begin{cases} 1 + x \; ; & \text{if } x \in I \cup A^c \\ -2 + x \; ; & \text{if } x \in A - I \; . \end{cases}$$

Then f is an H-function since the non-integral rationals are dense in the reals.

Now $f|A$ is not an H-function, since it fails to be so at every integral point. For let $n \in I$ and choose the neighbourhood $(n, n+2)$ of $f|A(n) = n+1$. Then $f|A^{-1}((n, n+2)) = (\{n\} \cup [n+2, n+4]) \cap A$ which is not a neighbourhood of n in the subspace A of X.

Again, it remains an open question whether the analogue of Theorem 2.2 holds for H-functions.

CHAPTER III

THE GRAPH TOPOLOGY AND COMPACT-OPEN TOPOLOGY

Before proceeding to a discussion of a comparison of the topology τ with known function space topologies, we include from Naimpally [12] the pertinent theorems concerning the comparison of the graph topology with the compact-open topology.

3.1 Definition: For each open set $U \subset X \times Y$ let $F_U = \{f \in \underline{F} | \Gamma(f) \subset U\}$. The graph topology Γ on $\underline{F}(X,Y)$ is that topology having as a base the family $\{F_U | U \text{ open in } X \times Y\}$.

It was shown above (Theorem 1.4) that p.c. $\subset \tau$, with no restrictions necessary on X or Y. A comparison of p.c. with Γ however requires that X be a T_1 space (i.e. a space in which singleton sets are closed).

A topological space is said to be T_1 if for each pair of distinct points x, y, there exist open sets U, V such that $x \in U$, $x \notin V$ and $y \in V$, $y \notin U$, which is equivalent to requiring singleton sets to be closed.

3.2 Theorem: If X is a T_1 space then p.c. $\subset \Gamma$ for $\underline{F}(X,Y)$.

Proof: If $W(a,U)$ is any subbase element for p.c., the set

$V = (X \times U) \cup (X-\{a\} \times Y)$ is open in $X \times Y$, and $F_V = W(a,U)$.

A subbase for the compact-open topology k on $\underline{F}(X,Y)$ consists of the family of sets of the form $W(K,U) = \{f \in \underline{F} | f(K) \subset U$, K compact in X, U open in $Y\}$. Since every singleton is a compact set it is clear that $p.c. \subset k$. In addition we have the following theorem.

<u>3.3 Theorem</u>: If X is a Hausdorff space, then $k \subset \Gamma$ on $\underline{F}(X,Y)$.

Proof: If $W(K,U)$ is a subbase element for k, then the set $V = (X \times U) \cup ((X-K) \times Y)$ is open in $X \times Y$, and $F_V = W(K,U)$.

<u>3.4 Theorem</u>: If X is compact and Hausdorff then $k = \Gamma$ on the space $\underline{C}(X,Y) \subset \underline{F}$.

Proof: Because of Theorem 3.3 we need only prove that $\Gamma \subset k$ on $\underline{C}(X,Y)$. Let M be an open subset of the product space $X \times Y$ and $f \in F_M$. For each $(x, f(x)) \in \Gamma(f)$ there exist open sets G_x and H_x such that $x \in G_x$, $f(x) \in H_x$ and $G_x \times H_x \subset M$. Since f is continuous there exists an open neighbourhood W_x of x such that $f(W_x) \subset H_x$. Let $U_x = W_x \cap G_x$. Since X is compact there exists an open neighbourhood V_x of x such that $\overline{V}_x \subset U_x$. Then $f(\overline{V}_x) \subset H_x$. The family $\{V_x | x \in X\}$ is an open cover of X, hence it admits a finite subcover $\{V_{x_i} \mid i = 1,2,\ldots,n\}$. Clearly we have $f \in \bigcap_{i=1}^{n} W(\overline{V}_{x_i}, H_{x_i})$ which is open in k. Furthermore, if $g \in \bigcap_{i=1}^{n} W(\overline{V}_{x_i}, H_{x_i})$ and $(x, g(x)) \in \Gamma(g)$ then $x \in \overline{V}_{x_i}$ for some i, so $g(x) \in H_{x_i}$. Therefore $(x, g(x)) \in \overline{V}_{x_i} \times H_{x_i} \subset M$, so that

$\Gamma(g) \subset M$. Thus $f \in \bigcap_{i=1}^{n} W(\overline{V}_{x_i}, H_{x_i}) \subset F_M$ so F_M is open in k , completing our proof.

In general, the inclusion $k \subset \Gamma$ is proper when X fails to be compact.

<u>3.5 Example</u>: Let $X = Y$ be the set of all real numbers with the usual topology. Let $f \in \underline{C}(X,Y)$ be defined by $f(x) = x$ for all $x \in X$. Let $\epsilon > 0$ be given, and let U be the open subset of $X \times Y$ consisting of the union of all ϵ-disks with centers at the points of $\Gamma(f)$. Then $\Gamma(f) \subset U$ so $f \in F_U$. Suppose $K_1, K_2, \ldots, K_n$ are compact subsets of X and $U_1, U_2, \ldots, U_n$ open subsets of Y such that $f \in \bigcap_{i=1}^{n} W(K_i, U_i)$. Since no finite union of compact sets covers X , there exists a real number $p \notin \bigcup_{i=1}^{n} K_i$ which is closed. Thus there exists $\delta > 0$ such that $A = \{x \in X / |x-p| \leq \delta\}$ is a subset of $(\bigcup_{i=1}^{n} K_i)^c$. Define the function $g: X \to Y$ as follows:

$$g(x) = \begin{cases} f(x) & \text{if } x \notin A . \\ \dfrac{2\epsilon + \delta}{\delta}(x-p+\delta) + p-\delta ; & \text{if } p-\delta \leq x < p \\ \dfrac{\delta - 2\epsilon}{\delta}(x-p-\delta) + p+\delta ; & \text{if } p \leq x \leq p+\delta \end{cases}$$

Then $g(p) = p + 2\epsilon$ and g is a continuous function satisfying $g \in \bigcap_{i=1}^{n} W(K_i, U_i)$. But $\Gamma(g) \not\subset U$ so $\bigcap_{i=1}^{n} W(K_i, U_i)$ is not a subset of F_U . Therefore k is properly contained in Γ .

The following theorem, due to Naimpally [12], supplies motivation for the definition of the graph topology Γ .

<u>3.6 Theorem</u>: The set $\underline{S}$ of all S-functions on X to Y is closed in $(\underline{F},\Gamma)$. Moreover $\underline{S} = \overline{\underline{C}}$.

Proof: Let $\{f_n | n \in D\}$ be a net in $\underline{S}$ such that $f_n \to f$ in the topology Γ . Let U be open in $X \times Y$ such that $f \in F_U$. There exists $n^* \in D$ such that $f_n \in F_U$ for $n \geq n^*$. Since f_{n^*} is an S-function, there exists a continuous function g such that $\Gamma(g) \subset U$. Thus $f \in \underline{S}$ so $\underline{S}$ is closed.

To prove the second assertion, we observe that since $\underline{C} \subset \underline{S}$ and $\underline{S}$ is closed, then $\overline{\underline{C}} \subset \underline{S}$. The reverse inclusion also obtains, since every Γ-neighbourhood of an S-function contains a continuous function, hence $\overline{\underline{C}} = \underline{S}$.

CHAPTER IV

COMPARISON OF FUNCTION SPACE TOPOLOGIES

We now discuss a comparison of the topology τ with the compact-open topology k, and the graph topology, Γ. It was found that meaningful results were possible only if the space X was assumed to be a total space (i.e. a space in which every open set is closed). Non-trivial total spaces do exist: consider for example the set N of positive integers, with the "Siamese-Twin" topology having as a base the collection $\underline{B} = \{\varphi, N, \{1,2\}, \{3,4\}, \{5,6\}, \ldots\}$.

4.1 Theorem: Let X be a T_1 space and Y a total space. Then the topology τ on $\underline{C}(X,Y)$ is contained in the graph topology.

Proof: Let $f \in \underline{C}(X,Y)$ and $x_o \in X$ with $f(x_o) \in U$ open in Y. We show the existence of an open subset V of $X \times Y$ such that $F_V = [f, x_o, U]$.

Let $V = [(X \times U) \cup (X - \{x_o\} \times Y] \cap [f^{-1}(U)^c \times \overline{U}]^c$. Then V is open in $X \times Y$ since $\{x_o\}$ is closed and, by continuity of f, so is $f^{-1}(U)^c$.

Let $g \in F_V$. Then $\underline{\Gamma(g)} \subset V$ so $(x_o, g(x_o)) \in X \times U$, and thus $g \in W(x_o, U)$. If $x \in g^{-1}(U) \subset g^{-1}(\overline{U})$ then $g(x) \in \overline{U}$. But $(x, g(x)) \in \Gamma(g)$ so $(x, g(x)) \notin f^{-1}(U)^c \times \overline{U}$. However, $g(x) \in \overline{U}$

so x must fail to be an element of $f^{-1}(U)^c$. Therefore $x \in f^{-1}(U) \subset \overline{f^{-1}(U)}$ proving that $\overline{g^{-1}(U)} \subset \overline{f^{-1}(U)}$. Therefore $g \in [f, x_o, U]$ so $F_V \subset [f, x_o, U]$.

To prove the reverse containment, let $g \in [f, x_o, U]$. Then $g \in W(x_o, U)$ so that $\Gamma(g) \subset (X \times U) \cup (X - \{x_o\} \times Y)$. For any $(x, g(x)) \in \Gamma(g)$ we consider two cases:

i) if $g(x) \in U = \overline{U}$ then $x \in g^{-1}(U) \subset \overline{g^{-1}(U)} \subset \overline{f^{-1}(U)} \subset f^{-1}(\overline{U})$ so $x \notin f^{-1}(U)^c$. Therefore $(x, g(x)) \notin [f^{-1}(U)^c \times \overline{U}]$ so it follows that $(x, g(x)) \in [f^{-1}(U)^c \times \overline{U}]^c$.

ii) if $g(x) \notin U$ then $g(x) \notin \overline{U}$ so that $(x, g(x)) \notin [f^{-1}(U)^c \times \overline{U}]$. Therefore $(x, g(x)) \in [f^{-1}(U)^c \times \overline{U}]^c$.

Therefore, in both cases $\Gamma(g) \subset [f^{-1}(U)^c \times \overline{U}]^c$ so $\Gamma(g) \subset V$. Therefore $[f, x_o, U] \subset F_V$ which completes the proof that $F_V = [f, x_o, U]$.

<u>4.2 Corollary</u>: If X is compact and Hausdorff, and if Y is a total space, then p.c. $\subset \tau \subset \Gamma = k$ on $\underline{C}(X, Y)$.

Proof: By Theorems 1.4, 4.1 and 3.3.

In the event that Y fails to be a total space, the topology τ is not comparable with either k or Γ ; for consider the following example:

<u>4.3 Example</u>: Let $X = Y = [0, 1]$ with the usual topology. (Then, since X is compact and Hausdorff we have $k = \Gamma$ on $\underline{C}(X, Y)$.) Define

the function $f:X \to Y$ by $f(x) = \frac{1}{2}$ for all $x \in X$. Let $\underline{0}$ be the open neighbourhood $(\frac{1}{4}, \frac{3}{4})$ of $\frac{1}{2}$, and $M = X \times \underline{0}$. Then $f \in F_M$ and we show that F_M is not open in τ. Suppose $x_i \in X$ and U_i open in Y have been chosen such that $f \in \bigcap_{i=1}^{n} [f, x_i, U_i]$. Assume without loss of generality that $0 \leq x_1 < x_2 < \ldots < x_n \leq 1$. Then $f(x_i) \in U_i$ for $i = 1, 2, \ldots, n$ so $\frac{1}{2} \in \bigcap_{i=1}^{n} U_i$. Furthermore $f^{-1}(U_i) = X$ for all i. Now $0 \notin \underline{0}$, so we define the function $g:X \to Y$ by

$$(4.3a) \qquad g(x) = \begin{cases} f(x) & \text{if } x \in X-(x_1, x_2) \\ \dfrac{2x-x_1-x_2}{2(x_1-x_2)} & \text{if } x \in (x_1, \dfrac{x_1+x_2}{2}] \\ \dfrac{2x-x_1-x_2}{2(x_2-x_1)} & \text{if } x \in (\dfrac{x_1+x_2}{2}, x_2) \end{cases} .$$

Then $g \in \underline{C}(X, Y)$ and $g(x_i) = f(x_i) \in U_i$. Furthermore $\overline{g^{-1}(U_i)} \subset X = \overline{f^{-1}(U_i)}$ for each i, so $g \in \bigcap_{i=1}^{n} [f, x_i, U_i]$. Now the point $(\frac{x_1+x_2}{2}, g(\frac{x_1+x_2}{2})) = (\frac{x_1+x_2}{2}, 0) \notin M$ so $g \notin F_M$. Thus no τ-neighbourhood of f is contained in F_M, hence F_M is not open in τ. Therefore $\Gamma \not\subset \tau$ and, since $k = \Gamma$ we have $k \not\subset \tau$.

We proceed to show that $\tau \not\subset \Gamma$. Let $f(x) = x$ and let $x_0 = \frac{1}{2}$. Let U be the neighbourhood $(\frac{1}{4}, \frac{3}{4})$ of $f(x_0)$. Then $[f, x_0, U]$ so a τ-neighbourhood of f. Suppose the open set $M \subset X \times Y$ has been chosen such that $f \in F_M$. Then the point $(\frac{1}{4}, \frac{1}{4}) \in M$ so we can choose $\epsilon > 0$ and $\epsilon < \frac{1}{4}$ such that $(\frac{1}{4}, \frac{1}{4}) \in (\frac{1}{4} - \epsilon, \frac{1}{4} + \epsilon) \times (\frac{1}{4} - \epsilon, \frac{1}{4} + \epsilon) \subset M$. Define the function $g: X \to Y$ as follows

$$(4.3b) \qquad g(x) = \begin{cases} f(x) & \text{for } x \leq \frac{1}{4} - \frac{\epsilon}{2} \text{ or } x \geq \frac{1}{4} \\ 4x + \frac{6\epsilon - 3}{4} & \text{for } \frac{1}{4} - \frac{\epsilon}{2} < x \leq \frac{1}{4} - \frac{\epsilon}{4} \\ -2x + \frac{3}{4} & \text{for } \frac{1}{4} - \frac{\epsilon}{4} < x < \frac{1}{4} \end{cases}$$

Then $g \in \underline{C}(X,Y)$, and clearly $\Gamma(g) \subset M$ since for all $x \in (\frac{1}{4} - \frac{\epsilon}{2}, \frac{1}{4})$ we have that $\frac{1}{4} - \frac{\epsilon}{2} < g(x) < \frac{1}{4} + \frac{\epsilon}{2}$ so $(x, g(x)) \in M$. Thus $g \in F_M$. But $\frac{1}{4} + \frac{\epsilon}{2} \in U$ and $g^{-1}(\frac{1}{4} + \frac{\epsilon}{2}) = \frac{1}{4} - \frac{\epsilon}{4} \notin \overline{f^{-1}(U)} = [\frac{1}{4}, \frac{3}{4}]$ since $\epsilon > 0$. Thus $\overline{g^{-1}(U)} \not\subset \overline{f^{-1}(U)}$ so $g \notin [f, x_o, U]$. Therefore no Γ neighbourhood of f is contained in $[f, x_o, U]$ so $[f, x_o, U]$ is not open in Γ. Therefore $\tau \not\subset \Gamma$ and hence $\tau \not\subset k$.

CHAPTER V

SET-OPEN TOPOLOGIES AND SIGMA TOPOLOGY

Arens and Dugundji [1] introduced a large class of topologies on $\underline{C}(X,Y) = Y^X$, called the set-open topologies, whose definition is patterned after the k topology, except that arbitrary families $\{A\}$ of subsets of X are admitted. A subclass of these topologies, called the sigma topologies, defined in terms of converings of X were also discussed, and members of both classes were characterized in terms of the notion of continuous convergence. It is the purpose of this chapter to investigate this method of classifying topologies on Y^X, and to determine conditions under which previously discussed topologies are subject to such classification.

It is recalled that a directed system $(\Delta, \geq)$ consists of a non-empty set $\Delta = \{\mu\}$ on which is defined a partial order $\geq$ satisfying

i) $\mu \geq \mu$

ii) if $\mu_1 \geq \mu_2$ and $\mu_2 \geq \mu_3$ then $\mu_1 \geq \mu_3$

iii) if $\mu_1, \mu_2 \in \Delta$, there exists $\mu_3 \in \Delta$ such that $\mu_3 \geq \mu_1$ and $\mu_3 \geq \mu_2$.

Every directed system $(\Delta, \geq)$ gives rise to a directed space $\Delta' = \Delta \cup \{\infty\}$ where $\infty \notin \Delta$, and the topology in Δ' is obtained by letting each $\{\mu\}$ be an open set, and letting neighbourhoods of ∞ be

sets of the form $(\mu') = \{\mu \in \Delta / \mu \geq \mu'\}$ for some $\mu' \in \Delta'$.

Suppose $\Omega = \{\omega\}$ is another directed system and $\Delta \times \Omega = \{(\mu,\omega) / \mu \in \Delta , \omega \in \Omega\}$. If we define $(\mu,\omega) \geq (\mu',\omega')$ whenever both $\mu \geq \mu'$ and $\omega \geq \omega'$ then $\Delta \times \Omega$ is also a directed sytem.

We recall further that a net or directed set in a space X is a function on a directed system Δ with values in X , denoted by $\{x_\mu\}_{\mu \in \Delta}$ or simply $\{x_\mu\}$ where no confusion can arise. A net $\{x_\mu\}$ converges to $x \in X$, written $x_\mu \to x$, if and only if for every open set G containing x , there exists $\mu' \in \Delta$ such that if $\mu \geq \mu'$ then $x_\mu \in G$. Furthermore, a function $f: X \to Y$ is continuous if and only if for every net $\{x_\mu\}$ in X , if $x_\mu \to x$ then $f(x_\mu) \to f(x)$.

We let Y^X denote the class of all continuous functions on a space (X,η) to a space (Y,ρ) . Without reference to a topology on Y^X we make the following definition

<u>5.1 Definition</u>: A net $\{f_\mu / \mu \in \Delta\}$ in Y^X is said to converge continuously to $f \in Y^X$ if and only if for every net $\{x_\omega / \omega \in \Omega\}$ in X which converges to $x \in X$, the net $\{f_\mu(x_\omega) / (\mu,\omega) \in \Delta \times \Omega\}$ converges in Y to $f(x)$.

The class $\{t\}$ of topologies on Y^X can be classified according as (i) convergence in $Y^X(t)$ implies continuous convergence, or (ii) continuous convergence implies convergence in $Y^X(t)$.

An alternate approach to the same classification is to consider a third space (Z,λ) and a function $g: Z \times X \to Y$. We can define an associated function $g^*: Z \to Y^X$ by letting $g^*(z)(x) = g(z,x)$. Then

a topology t for Y^X may be such that for any space Z , (iii) if g is continuous then g^* is continuous, or (iv) if g^* is continuous then g is continuous.

It will be shown below that (i) is equivalent to (iii) and that (ii) is equivalent to (iv). We make the following definition.

<u>5.2 Definition</u>: A topology t for Y^X is said to be proper if for any space (Z,λ) the continuity of $g:Z \times X \to Y$ implies the continuity of $g^*:Z \to Y^X(t)$ where $g^*(z)(x) = g(z,x)$. A topology t for Y^X is said to be admissible if for any space (Z,λ) the continuity of $g^*:Z \to Y^X(t)$ implies the continuity of $g:Z \times X \to Y$, where $g(z,x) = g^*(z)(x)$.

<u>5.3 Theorem:</u> A topology t for Y^X is admissible if and only if the mapping $w:X \times Y^X(t) \to Y$ defined by $w(x,f) = f(x)$ (called the evaluation map) is continuous jointly in x and f .

Proof: Let t be admissible for Y^X . Choose $Z = Y^X(t)$ and let $i^*:Y^X(t) \to Y^X(t)$ be the identity map. Since i^* is continuous, then by Definition 5.2 so is the mapping i where $i:Y^X(t) \times X \to Y$ is defined by $i(f,x) = i^*(f)(x)$. But $i^*(f)(x) = f(x)$ since i^* is the identity map on $Y^X(t)$. Therefore the mapping $w:(x,f) \to f(x)$ is also continuous.

Now suppose $w:X \times Y^X(t) \to Y$ is continuous and let $g^*:Z \to Y^X(t)$ be continuous. Define $h:X \times Z \to X \times Y^X(t)$ by $h(x,z) = (x,g^*(z))$. Then h is continuous since g^* is continuous. Now $w(h(x,z)) = w(x,g^*(z)) = g^*(z)(x)$ so wh is the associated map for g^* , namely

g itself. Since w and h are continuous, so also is $g = wh$. Therefore t is admissible.

5.4 Theorem: A net $\{f_\mu\}$ in Y^X converges continuously to $f \in Y^X$ if and only if for every neighbourhood W of $f(x)$ there exists $\mu' \in \Delta$ and a neighbourhood V of x such that if $\mu \geq \mu'$ then $f_\mu(V) \subset W$.

Proof: Suppose $\{f_\mu\}$ converges continuously to f, and let W be any neighbourhood of $f(x)$. Let $\{N_\omega | \omega \in \Omega\}$ be the system of neighbourhoods of x, directed by set inclusion. For each $\omega \in \Omega$ choose $x_\omega \in N_\omega$, then the net $\{x_\omega\}$ converges to the point x. Since $\{f_\mu\}$ converges continuously to f, then given $\omega \in \Omega$ there exists $\mu' \in \Delta$ such that if $\mu \geq \mu'$ then $f_\mu(x_\omega) \in W$. Since the choice of x_ω was arbitrary, then $f_\mu(x) \in W$ for all $x \in N_\omega$ if $\mu \geq \mu'$. Let $V = N_\omega$ so that $f_\mu(V) \subset W$ for $\mu \geq \mu'$.

Now, suppose such a μ' and V exist for every neighbourhood W of $f(x)$. Let $\{x_\omega | \omega \in \Omega\}$ be a net in X such that $x_\omega \to x$. There exists a $\mu' \in \Delta$ and a neighbourhood V of x such that if $\mu \geq \mu'$ then $f_\mu(V) \subset W$, and there exists $\omega' \in \Omega$ such that if $\omega \geq \omega'$ then $x_\omega \in V$. Let $(\mu, \omega) \geq (\mu', \omega')$. Then $x_\omega \in V$ and $f_\mu(x_\omega) \in f_\mu(V) \subset W$ so $f_\mu(x_\omega) \in W$, proving continuous convergence.

5.5 Theorem: A topology t for Y^X is proper if and only if for every directed system $(\Delta, \geq)$ and every net $\{f_\mu | \mu \in \Delta\}$ in Y^X, if f_μ converges continuously to $f \in Y^X$, then f_μ is t-convergent to f.

Proof: Suppose t is proper for Y^X and let $\{f_\mu | \mu \in \Delta\}$ converge continuously to $f \in Y^X$. Let $\Delta' = \Delta \cup \{\infty\}$ be the directed space obtained from Δ, and define $g: \Delta' \times X \to Y$ by $g(\mu, x) = f_\mu(x)$ and $g(\infty, x) = f(x)$. By Theorem 5.4 g is continuous, since if $f(x) \in W$ open, then there exists $\mu' \in \Delta$ and an open neighbourhood V of x such that $(\infty, x) \in (\mu') \times V$, $g((\mu') \times V) = \{f_\mu(x) | \mu \geq \mu', x \in V\} \subset W$; similarly if $f_\mu(x) \in W$ open, then there exists a neighbourhood V of x such that $f_\mu(V) \subset W$ thus $(\mu, x) \in \{\mu\} \times V$ and $g(\{\mu\} \times V) = f_\mu(V) \subset W$. Now since g is continuous then by hypotheses so is $g^*: \Delta' \to Y^X(t)$. But the identity function on Δ to Δ' is a net in Δ' which converges to ∞. Thus $g^*(\mu)$ converges to $g^*(\infty)$, or $f_\mu = g^*(\mu) \to g^*(\infty) = f$. Thus f_μ converges in t to f as desired.

Conversely, suppose continuous convergence implies t-convergence, and let $g: Z \times X \to Y$ be continuous. Let $\{z_\mu | \mu \in \Delta\}$ be a net in Z converging to $z \in Z$. Then for each z_μ, the function $g^*(z_\mu) \in Y^X$ so is continuous. Thus $\{g^*(z_\mu) | \mu \in \Delta\}$ is a net in Y^X. Let $\{x_\omega | \omega \in \Omega\}$ be a net in X converging to $x \in X$, and let W be an open neighbourhood in Y of $g^*(z)(x)$. By continuity of $g^*(z)$, there exists $\omega' \in \Omega$ such that if $\omega \geq \omega'$ then $g^*(z)(x_\omega) \in W$. Thus there exists $\mu' \in \Delta$ such that if $\mu \geq \mu'$ then $g^*(z_\mu)(x_\omega) \in W$ whenever $g^*(z)(x_\omega) \in W$. Let $(\mu, \omega) \geq (\mu', \omega')$. Then $g^*(z_\mu)(x_\omega) \in W$ so $g^*(z_\mu)$ converges continuously to $g^*(z)$. By hypotheses, then $g^*(z_\mu)$ is t-convergent to $g^*(z)$ so $g^*(z) \in Y^X(t)$. That is, $g^*(z)$ is continuous. Therefore, continuity of g implies the continuity of g^* so t is proper.

<u>5.6 Theorem</u>: A topology t on Y^X is admissible if and only if for every net $\{f_\mu | \mu \in \Delta\}$ in Y^X, the convergence of f_μ to f in the space $Y^X(t)$ implies continuous convergence of f_μ to f.

Proof: Suppose t is admissible for Y^X. Let $\{f_\mu | \mu \in \Delta\}$ be a net in Y^X such that $\{f_\mu\}$ is t-convergent to $f \in Y^X$. Let $\{x_\omega | \omega \in \Omega\}$ be a net in X which converges to $x \in X$ and let W be an open neighbourhood of $f(x)$. Define $w: X \times Y^X \to Y$ by $w(x,f) = f(x)$. By Theorem 5.3 w is continuous, hence there are neighbourhoods V of x and F of f such that $w(V \times F) \subset W$. There exists $(\omega', \mu') \in \Omega \times \Delta$ such that if $(\omega, \mu) \geq (\omega', \mu')$ then $f_\mu \in F$ and $x_\omega \in V$. Thus $(x_\omega, f_\mu) \in V \times F$ so $w(x_\omega, f_\mu) = f_\mu(x_\omega) \in w(V \times F) \subset W$. Therefore $\{f_\mu\}$ converges continuously to f.

Now, suppose t-convergence implies continuous convergence, and let Z be any space and $g^*: Z \to Y^X(t)$ be continuous. Let $\{(z_\mu, x_\omega) | \mu \in \Delta, \omega \in \Omega\}$ be a net in $Z \times X$ which converges to $(z,x) \in Z \times X$. Since projection maps are continuous, the net $\{z_\mu\}$ converges to z and $\{x_\omega\}$ converges to x. By the continuity of g^*, the net $\{g^*(z_\mu)\}$ converges in t to $g^*(z)$. But t-convergence implies continuous convergence, hence $\{g^*(z_\mu)(x_\omega)\}$ converges in Y to $g^*(z)(x)$. But $g^*(z_\mu)(x_\omega) = g(z_\mu, x_\omega)$ so the net $\{g(z_\mu, x_\omega)\}$ converges in Y to $g^*(z)(x) = g(z,x)$. Therefore g is continuous so by Definition 5.2 t is admissible for Y^X.

Since we are concerned with various topologies on the set Y^X, it is convenient to introduce the following definition:

5.7 Definition: Let t and u be two topologies on a set E. We say $t \leq u$ (or $u \geq t$) if for every $V \in t$ we have $V \in u$.

In the above definition, the relation $\leq$ defines a partial ordering for the class of all topologies on a set E. Birkhoff [2], p. 173.

5.8 Theorem: Let t and u be two topologies for a set A. Then $t \leq u$ if and only if the identity map $i:(A,u) \rightarrow (A,t)$ is continuous.

Proof: The identity map is continuous if and only if every element of t is also an element of u.

5.9 Theorem: Let t and u be two topologies for Y^X. Then

i) if t is admissible and $t \leq u$, then u is admissible.

ii) if u is proper and $t \leq u$ then t is proper.

Proof: (i) Let t be admissible and $t \leq u$. Let V be open in Y. Now $w:X \times Y^X(t) \rightarrow Y$ defined by $w(x,f) = f(x)$ is continuous so $w^{-1}(V)$ is open in $X \times Y^X(t)$. But $t \leq u$ so $w^{-1}(V)$ is also open in $X \times Y^X(u)$. Therefore the map $w:X \times Y^X(u) \rightarrow Y$ is also continuous so u is admissible.

(ii) Let u be proper, $t \leq u$ and let $g:Z \times X \rightarrow Y$ be continuous. Then, since u is proper, $g^*:Z \rightarrow Y^X(u)$ is continuous. If $V \in t$ then $V \in u$ so $g^{*-1}(V)$ is open in Z, therefore

$g^*: Z \to Y^X(t)$ is continuous so that t is proper.

<u>5.10 Theorem</u>: If t, u are two topologies for Y^X such that t is proper and u is admissible, then $t \leq u$.

Proof: Since u is admissible the mapping $w: X \times Y^X(u) \to Y$ is continuous. Therefore, so is the mapping $\bar{w}: Y^X(u) \times X \to Y$ defined by $\bar{w}(f, x) = w(x, f) = f(x)$. Since t is proper, the associated mapping for $\bar{w}$, $\bar{w}^*: Y^X(u) \to Y^X(t)$, is continuous. But $\bar{w}^*$ is the identity map since for all $x \in X$, $\bar{w}^* f(x) = \bar{w}(f, x) = w(x, f) = f(x)$ for all $f \in Y^X$, therefore $\bar{w}^*(f) = f$. Thus by Theorem 5.8 $t \leq u$.

<u>5.11 Theorem</u>:

i) The discrete topology on Y^X is the largest admissible topology.

ii) The trivial topology on Y^X is the smallest proper topology.

Proof: (i) Let d be the discrete topology on Y^X, which is certainly the largest. It is also admissible since $w: X \times Y^X(d) \to Y$ is continuous.

(ii) Let t be the trivial topology on Y^X, which is certainly the smallest. Let Z be any space and let $g: Z \times X \to Y$ be continuous. Then $g^*: Z \to Y^X(t)$ is certainly continuous so t is proper.

We now define a class of topologies for Y^X which can be classified according to the above discussed material, and investigate their relationships to previously discussed topologies. Let $\underline{S}$ be an arbitrary cover of X by open sets, and keep $\underline{S}$ fixed throughout the following discussion. Let F be any closed set in X which is contained in some member of $\underline{S}$, and let V be an open set in Y. We define $(F,V) = \{g \in Y^X | g(F) \subset V\}$, and take the family of all sets (F,V) as a subbase for a topology on Y^X.

5.12 Definition: The topology on Y^X thus determined by $\underline{S}$ is called the sigma topology, and will be denoted by σ.

A topological space is said to be regular if for every closed set A and $x \notin A$, there exist disjoint open neighbourhoods of x and A.

5.13 Theorem: Let X be regular and Y arbitrary. For any $\underline{S}$, the sigma topology, σ, on Y^X is always admissible.

Proof: Let $f \in Y^X$ and $x \in X$ such that $f(x)$ is an element of an open set $W \subset Y$. Since f is continuous there is an open set U containing x such that $f(U) \subset W$. Now there exists an element $S \in \underline{S}$ such that $x \in S$, and $S \cap U$ is an open neighbourhood of x such that $f(S \cap U) \subset W$. Since X is regular there exists an open neighbourhood V of x such that $\overline{V} \subset S \cap U \subset S$ and $f(\overline{V}) \subset W$. Now we have $f \in (\overline{V},W) \in \sigma$ and $x \in V$ so $V \times (\overline{V},W)$ is an open neighbourhood of (x,f) in $X \times Y^X(\sigma)$. Furthermore, $w:(V \times (\overline{V},W)) = \{g(t) | t \in V, g \in (\overline{V},W)\} \subset W$ so the evaluation map $w:X \times Y^X(\sigma) \to Y$ is continuous,

hence σ is admissible.

5.14 Theorem: If X is T_1 then p.c. $\subset \sigma$.

Proof: Let $W(x,U)$ be any subbase element for p.c. Since X is T_1 then $\{x\}$ is closed and furthermore, $\{x\} \subset S$ for some $S \in \underline{S}$ since $\underline{S}$ is a cover of X . Therefore $W(x,U) = (\{x\},U) \in \sigma$.

5.15 Theorem: $\sigma \subset \Gamma$.

Proof: Let A be closed in X such that $A \subset S$ for some $S \in \underline{S}$. Let U be open in Y . Then $V = (X \times U) \cup (X-A \times Y)$ is open in $X \times Y$ and $F_V = (A,U)$. Therefore (A,U) is open in Γ so $\sigma \subset \Gamma$.

5.16 Corollary: If X is T_1 then p.c. $\subset \sigma \subset \Gamma$.

5.17 Theorem: If X is compact then $\sigma \subset k$.

Proof: For any closed set A such that $A \subset S \in \underline{S}$ and U open in Y , A is compact, and therefore $(A,U) = W(A,U)$ so $\sigma \subset k$.

5.18 Corollary: If X is compact and T_1 then p.c. $\subset \sigma \subset k$.

We now show by counter-example that in general the topology τ on Y^X is not comparable with a sigma topology. Example 4.3 shows that in general τ is not contained in Γ , hence by Theorem 5.15 it is not contained in σ . We now show that in general σ is not contained in τ .

<u>5.19 Example</u>: Let X, Y be as in 4.3, i.e. $X = Y = [0,1]$ with usual topology. Let $\underline{S}$ be the open cover of X consisting of all sets of the form $[0,a) \cup (b,1]$ where a, b are rational and $a,b \in (0,1)$. Define $f(x) = \frac{1}{2}$ for all $x \in X$ and let A be the closed interval $[\frac{1}{4}, \frac{3}{4}]$ with U the open interval $(\frac{1}{8}, \frac{7}{8})$. Then $f \in (A,U) \in \sigma$ since A is contained in the element $[0, \frac{1}{8}) \cup (\frac{1}{16}, 1]$ of S. Now suppose $x_i \in X$ and U_i open in Y have been chosen such that $f \in \bigcap_{i=1}^{n} [f, x_i, U_i]$. Then $\frac{1}{2} \in U_i$ for all i. Assume without loss of generality that $0 \leq x_1 < x_2 < x_3 < \dots < x_n \leq 1$. Now there exists $a \in A$ such that $a \neq x_i$ for all $i = 1,2,\dots,n$, so there exists k such that $x_k < a < x_{k+1}$. Define the function g as follows:

$$g(x) = \begin{cases} f(x) & \text{if} \quad x \in X - (x_k, x_{k+1}) \\ \dfrac{x-a}{2(x_k-a)} & \text{if} \quad x \in [x_k, a] \\ \dfrac{x-a}{2(x_{k+1}-a)} & \text{if} \quad x \in (a, x_{k+1}) \end{cases}$$

Then $g \in Y^X$ and $g(x_i) = f(x_i)$ for all i. Furthermore $\overline{g^{-1}(U_i)} \subset X = \overline{f^{-1}(U_i)}$ and thus $g \in \bigcap_{i=1}^{n} [f, x_i, U_i]$. But clearly $g(A) \not\subset U$ since $g(a) = 0$. Therefore $g \notin (A,U)$ so (A,U) contains no τ-open neighbourhood of f. Therefore (A,U) is not open in τ so $\sigma \not\subset \tau$.

A relatively large class of topologies for Y^X, including the class of sigma topologies, can be introduced by considering any family $\{A\}$ of subsets of X, and taking the family of sets (A,U) for $A \in \{A\}$, U open in Y, as a subbase in Y^X.

<u>5.20 Definition</u>: The topology described above is called the $\{A\}$-open topology, or set-open topology induced by $\{A\}$, and will be denoted by S. The symbol $Y^X(S)$ will denote the set Y^X endowed with the topology S.

<u>5.21 Theorem</u>: If all the sets in $\{A\}$ are compact, then the $\{A\}$-open topology S is proper for Y^X.

Proof: Suppose Z is any space and suppose $g:Z \times X \to Y$ is continuous. To prove that $g^*:Z \to Y^X(S)$ is continuous it suffices to show that given any subbase set (A,V) containing $g^*(z_o)$, there is a neighbourhood U of z_o with $g^*(U) \subset (A,V)$. Let $g^*(z_o) \in (A,V)$. Then $g^*(z_o)(A) \subset V$. But $g^*(z_o)(A) = \{g^*(z_o)(a) | a \in A\} = \{g(z_o,a) | a \in A\} = g(z_o \times A)$. Thus $z_o \times A \subset g^{-1}(V)$ which is open by continuity of g. For each $(z_o,a) \in z_o \times A$ there exists an open set $W_a \subset Z \times X$ such that $(z_o,a) \in W_a \subset g^{-1}(V)$. The collection $\{W_a | a \in A\}$ is an open cover of $z_o \times A$, which is compact, so there exists a finite subcover of $z_o \times A$, say $W_{a_1}, W_{a_2}, \ldots, W_{a_n}$. The projection maps of $Z \times X$ into Z and X respectively are continuous and open, so the projection on Z of each W_{a_i} is an open neighbourhood of z_o: hence so is the set U obtained by taking the intersection of all such projections. Then $z_o \times A \subset U \times A \subset \bigcup_{i=1}^{n} W_{a_i} \subset g^{-1}(V)$. Now $g^*(U) = \{g^*(z) | z \in U\}$, but for any $z \in U$ we have $g^*(z)(A) = g(z \times A) \subset g(U \times A) \subset gg^{-1}(V) \subset V$. Thus $g^*(z) \in (A,V)$ for each $z \in U$, hence $g^*(U) \subset (A,V)$ as required. So g^* is continuous, proving that S is proper for Y^X.

<u>5.22 Corollary</u>: The k topology on Y^X is proper.

Proof: The k topology is that set-open topology based on the family $\{K\}$ of all compact subsets of X .

We have shown above that a proper topology for Y^X is smaller than any admissible topology. (Theorem 5.10.) We thus have the following result:

<u>5.23 Theorem</u>: If X is compact and regular, and σ is any sigma topology for Y^X , then $k = \sigma$.

Proof: By 5.22 k is proper for Y^X and by 5.17 $\sigma \subset k$. But σ is admissible by 5.13 and hence by 5.10 $k \subset \sigma$. Therefore $k = \sigma$.

<u>5.24 Theorem</u>: If X is regular then Γ is admissible for Y^X .

Proof: By 5.13 σ is admissible, and by 5.15 $\sigma \subset \Gamma$. Thus by 5.9 Γ is admissible.

<u>5.25 Theorem</u>: p.c. is proper for Y^X .

Proof: By 5.22 k is proper, and p.c. $\subset k$, thus by 5.9 p.c. is proper.

We are now able to augment the conditions under which the pointwise convergence topology is contained in the graph topology on Y^X .

This is known to hold, by 3.2, when X is a T_1 space. However, it is well-known that there do exist spaces which are regular but fail to be T_1. Thus the following theorem is of some interest.

<u>5.26 Theorem</u>: If X is regular then p.c. $\subset \Gamma$ on Y^X.

Proof: By 5.24 Γ is admissible and by 5.25 p.c. is proper for Y^X. Thus by 5.10 p.c. $\subset \Gamma$.

We conclude this chapter with a few comments concerning the characterization of the topology τ by the methods of this chapter.

We have seen already, (Example 4.3) that even under very strong conditions on X and Y, the k topology is not comparable with τ. Thus we can say that under even these strong conditions, τ is not admissible for Y^X, for if it were, then by Theorems 5.22 and 5.10 we obtain the contradiction $k \subset \tau$.

Similarly, Example 4.3 shows that under very strong conditions τ is not comparable with Γ. Thus we can say that even under the conditions present in Example 4.3, τ is not a proper topology for Y^X, for if it were, then by Theorems 5.24 and 5.10 we obtain the contradiction that $\tau \subset \Gamma$.

As a result, we cannot expect that even under strong conditions on X and Y the topology τ can be classified by the methods described in the present chapter.

CHAPTER VI

SEPARATION AXIOMS

In this chapter we shall be concerned with the conditions under which the function space $(\underline{F},\tau)$ satisfies certain separation axioms of general topology.

A topological space is said to be a T_o space if for each pair of distinct points there is an open set containing one but not the other.

6.1 Theorem: $(\underline{F},\tau)$ is a T_o space if and only if (Y,ρ) is T_o .

Proof: The T_o property is both hereditary and productive, and by Theorem 1.4 the pointwise convergence topology is contained in τ . Therefore, if Y is T_o then so is $(\underline{F}, \text{p.c.})$ and hence $(\underline{F},\tau)$. So we prove only that Y is T_o whenever $(\underline{F},\tau)$ is T_o .

Suppose $(\underline{F},\tau)$ is T_o and let $a,b \in Y$ such that $a \neq b$. Define f , and g to be the functions $f(x) = a$ and $g(x) = b$ for all $x \in X$. Then $f \neq g$ and since $(\underline{F},\tau)$ is T_o there exists an open set $U \in \tau$ such that, without loss of generality, $f \in U$ and $g \notin U$. Let $x_i \in X$ and U_i open in Y be chosen such that $f \in \bigcap_{i=1}^{n} [f,x_i,U_i] \subset U$. Then $a \in U_i$ for all $i = 1,2,\ldots,n$, so

$a \in \bigcap_{i=1}^{n} U_i$ which is an open set. Now $g \notin U$ so $g \notin \bigcap_{i=1}^{n} [f, x_i, U_i]$.
But $\underline{g^{-1}(U_i)} \subset X = \underline{f^{-1}(U_i)}$ for each i, hence there must exist a k such that $g(x_k) \notin U_k$. Then $b \notin U_k$ so that $b \notin \bigcap_{i=1}^{n} U_i$. But $a \in \bigcap_{i=1}^{n} U_i$ so Y is a T_o space.

<u>6.2 Theorem</u>: $(\underline{F}, \tau)$ is a T_1 space if and only if Y is a T_1 space.

Proof: The T_1 property is hereditary and productive, therefore by Theorem 1.4, if Y is T_1 so is $(\underline{F}, \tau)$.

Suppose $(\underline{F}, \tau)$ is T_1 and $a, b \in Y$ such that $a \neq b$. Define $f(x) = a$ and $g(x) = b$ for all $x \in X$. There exist open sets U, V in $\underline{F}$ with $f \in U$, $g \in V$, $f \notin V$ and $g \notin U$. Let $x_i \in X$ and U_i open in Y be chosen such that $f = \bigcap_{i=1}^{n} [f, x_i, U_i] \subset U$ and let $y_i \in X$ and V_i open in Y be chosen such that $g \in \bigcap_{i=1}^{m} [g, y_i, V_i] \subset V$. Now $g \notin \bigcap_{i=1}^{n} [f, x_i, U_i]$ but $\underline{g^{-1}(U_i)} \subset X = \underline{f^{-1}(U_i)}$ so there exists a k such that $g \notin W(x_k, U_k)$. Similarly, there exists a j such that $f \notin W(y_j, V_j)$. Therefore $b = g(x_k) \notin U_k$ and $a = f(x_k) \in U_k$, similarly $a = f(y_j) \notin V_j$ and $b = g(y_j) \in V_j$. Therefore U_k and V_j are open sets in Y containing a and b respectively, but $a \notin V_j$ and $b \notin U_k$. Thus Y is a T_1 space.

<u>6.3 Theorem</u>: $(\underline{F}, \tau)$ is T_2 if and only if Y is T_2.

Proof: The T_2 property is hereditary and productive, therefore by Theorem 1.4, if Y is T_2 so is $(\underline{F}, \tau)$.

Suppose (F,τ) is T_2 and let $a,b \in Y$ such that $a \neq b$. Define $f(x) = a$ and $g(x) = b$ for all $x \in X$. Since $f \neq g$, f, g have disjoint open neighbourhoods U and V. Let $x_i, y_i \in X$ and U_i, V_i open in Y be chosen such that $f \in \bigcap_{i=1}^{n}[f,x_i,U_i] \subset U$ $g \in \bigcap_{i=1}^{m}[g,y_i,V_i] \subset V$. Then $(\bigcap_{i=1}^{n}[f,x_i,U_i]) \cap (\bigcap_{i=1}^{m}[g,y_i,V_i]) = \phi$ since $U \cap V = \phi$. Furthermore $a \in \bigcap_{i=1}^{n}U_i$ and $b \in \bigcap_{i=1}^{m}V_i$, both sets being open in Y. We claim that $(\bigcap_{i=1}^{n}U_i) \cap (\bigcap_{i=1}^{m}V_i) = \phi$, for if not, it would contain a point y : then the function defined by $h(x) = y$ for all $x \in X$ would satisfy $h(x_i) \in U_i$ for $i = 1,2,\ldots,n$ and $h(y_i) \in V_i$ for $i = 1,2,\ldots,m$. Furthermore h would satisfy $\overline{h^{-1}(U_i)} \subset X = \overline{f^{-1}(U_i)}$ and $\overline{h^{-1}(V_i)} \subset X = \overline{g^{-1}(V_i)}$ so that $h \in U \cap V$ which is impossible. Therefore Y is a T_2 space.

The author was unable to determine under what non-trivial conditions the properties of regularity and complete regularity are induced in $(\underline{F},\tau)$, and unsuccessful in attempts to construct a counter-example. It remains an open question, therefore, whether in general a uniform structure can be defined on $\underline{F}$ which is compatible with τ.

Furthermore, no satisfactory answer was found to the question of metrizability of $(\underline{F},\tau)$.

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